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# A Trigonometric Quadrature Rule for Cauchy Integrals with Jacobi Weight

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In this paper, we consider a quadrature rule for Cauchy integrals of the form $I(wf; s) = \fint_{-1}^{1} w(t)\, f(t)/(t-s)\, dt$, $-1 < s < 1$, for a smooth density function $f(t)$ and Jacobi's weights $w(t) = (1-t)^{\alpha}\,(1+t)^{\beta}$, $\alpha, \beta > -1/2$. Using the change of variables $t = \cos y$, $s = \cos x$ and subtracting out the singularity, we propose a trigonometric quadrature rule. We obtain the error bounds independent of the set of values of poles and construct an automatic quadrature of nonadaptive type. © 2001 Academic Press

*Key Words:* Cauchy integral; quadrature rule; trigonometric interpolation; Jacobi weight.

## 1. INTRODUCTION

In the recent literature, the numerical evaluation for the Cauchy principal value (CPV) integral has received considerable attention because of its importance in many problems of mathematical physics which can be formulated as Cauchy singular integral equations. There are two basic approaches for numerically approximating CPV integrals. The majority of numerical methods are based on orthogonal polynomial approximations [2, 4, 15, 19], while the other methods make use of piecewise polynomial approximations [3, 6, 9, 12, 13]. It is well known that the former method converges very fast for differentiable density functions (see [6]). To take full advantage of the orthogonal polynomial approximation, the collocation points should be fixed at the zeros of the orthogonal polynomials. This fact reduces the flexibility of the orthogonal polynomial approximations, especially when the problem contains a system of integral equations defined over several different intervals (see [3]). On the other hand, the piecewise polynomial approximation does not have this restriction on the location of

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the collocation points. However, their implementation suffers from a lack of efficient numerical algorithms for the computation of the quadrature weights (see [6, 12]).

Hasegawa and Torii [7, 8] have developed automatic quadrature formulae for CPV integrals and hypersingular integrals with the Legendre weight function. They obtained the error estimate independent of the set of values of poles. The rules of [7, 8], however, need the extra evaluation of the density function and its derivatives for the set of values of poles.

We recently proposed an automatic quadrature based on trigonometric interpolation (see [11]) for CPV integrals with the Legendre weight function. In [10], we also extend the approaches of [11] to CPV integrals with the ultraspherical weight function $w(t) = (1-t^2)^{-1/2+\lambda}$, $\lambda > -1/2$, which yields an error estimate independent of the set of values of poles.

In this paper, we develop a trigonometric quadrature for CPV integrals with a general Jacobi's weight function $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$, $\alpha, \beta > -1/2$. The proposed rule yields a stable algorithm for evaluating the quadrature weights and the error bounds independent of the set of values of poles, which enables us to construct an automatic quadrature of a nonadaptive type.

The rest of the paper is organized as follows. In Section 2, we reformulate CPV integrals as standard ones using the cosine transformation. Section 3 describes trigonometric interpolation and gives the trigonometric quadrature for the CPV integral with Jacobi's weight. In Section 4, we estimate the asymptotic behavior of the quadrature weights. The error estimate is discussed in Section 5.

## 2. STATEMENT OF THE PROBLEM

In this paper we consider the numerical evaluation of CPV integrals of the form

$$
\begin{aligned}
I(wf; s) &= ⨍_{-1}^{1} w(t) \frac{f(t)}{t-s}\, dt \\
&= \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{s-\varepsilon} + \int_{s+\varepsilon}^{1} \right\} w(t) \frac{f(t)}{t-s}\, dt, \qquad |s| < 1, \qquad (2.1)
\end{aligned}
$$

where $f(t)$ is assumed to be smooth on $[-1, 1]$ and $w(t)$ is a weight function given by

$$
w(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad (2.2)
$$

where $\alpha, \beta > -1/2$.

To begin we make use of the change of variables $t = \cos y$ and $s = \cos x$ in (2.1) and then obtain

$$I(w(\cos\,\cdot\,)\; f(\cos\,\cdot\,);\cos x) = ⨍_0^\pi w(\cos y) \frac{f(\cos y)\sin y}{\cos y - \cos x}\, dy$$

$$= 2^{\alpha+\beta+1} \int_0^\pi \sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2} \frac{h(y) - h(x)}{\cos y - \cos x}\, dy$$

$$+ h(x)\, q_0(x)\,, \qquad x \in (0, \pi), \tag{2.3}$$

where $h(y) = f(\cos y)$, $\bar{\alpha} = 2\alpha + 1$, $\bar{\beta} = 2\beta + 1$, and

$$q_0(x) = 2^{\alpha+\beta+1} ⨍_0^\pi \frac{\sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2}}{\cos y - \cos x}\, dy\,.$$

We rewrite Eq. (2.3) as

$$I(w(\cos\,\cdot\,)\, h; \cos x) = 2^{\alpha+\beta+1} Q(h; x) + h(x)\, q_0(x)\,, \tag{2.4}$$

where $h(y) = f(\cos y)$ and

$$Q(h; x) = \int_0^\pi \sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2} \frac{h(y) - h(x)}{\cos y - \cos x}\, dy\,.$$

For $\alpha, \beta > -1$, $q_0(x)$ satisfies

$$q_0(x) = \pi \cot(\pi\alpha)\, w(\cos x)$$
$$- 2^{\alpha+\beta} \frac{\Gamma(\alpha)\, \Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} F\left(1, -\alpha-\beta; 1-\alpha; \frac{1-\cos x}{2}\right), \tag{2.5}$$

where $F(\,\cdot\,, \,\cdot\,; \,\cdot\,, \,\cdot\,)$ is the hypergeometric function and $\Gamma(z)$ the gamma function (see, e.g., [18]). The problem of evaluating the CPV integral $I(wf; s)$ of (2.1) is now reduced to evaluating the integrals $Q(h; x)$ of (2.4).

## 3. TRIGONOMETRIC INTERPOLATION METHODS

Following the procedure of [10, 11], we approximate $h(y)$ of (2.4) by the sum of $\cos ky$,

$$p_N(y) = \sum_{k=0}^{N}{}'' a_k^N \cos ky\,, \qquad 0 \leqslant y \leqslant \pi\,, \tag{3.1}$$

where $a_k^N$ are determined to satisfy the interpolation conditions

$$h(\pi j/N) = p_N(\pi j/N), \qquad 0 \leqslant j \leqslant N \tag{3.2}$$

and given as follows [1, 10, 11]:

$$a_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' h(\pi j/N) \cos(\pi jk/N), \qquad 0 \leqslant k \leqslant N. \tag{3.3}$$

Here the coefficients $a_k^N$ can be computed by using the fast cosine transform (FCT) [5]. The double prime of (3.1) denotes the summation whose first and last terms are halved. We now approximate $h(y)$ and $h(x)$ of (2.4) by $p_N(y)$ and $p_N(x)$ of (3.1), respectively, and then obtain the trigonometric quadrature rule for $Q(h; x)$

$$\begin{aligned} Q_N(h; x) &= Q(p_N; x) \\ &= \sum_{k=0}^{N}{}'' a_k^N J_k(x), \end{aligned} \tag{3.4}$$

where

$$J_k(x) = \int_0^\pi \sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2} \frac{\cos ky - \cos kx}{\cos y - \cos x} \, dy, \qquad k = 0, 1, \ldots . \tag{3.5}$$

Then, we clearly see that

$$J_0(x) = 0.$$

By the definition of the beta function (see, e.g., [16]), we also see that

$$\begin{aligned} J_1(x) &= 2 \int_0^{\pi/2} \sin^{2(\alpha+1)-1} y \cos^{2(\beta+1)-1} y \, dy \\ &= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \end{aligned}$$

From the following recurrence relation

$$\begin{aligned} &\frac{\cos(k+2)\,y - \cos(k+2)\,x}{\cos y - \cos x} + \frac{\cos ky - \cos kx}{\cos y - \cos x} \\ &\quad = 2 \cos x \frac{\cos(k+1)\,y - \cos(k+1)\,x}{\cos y - \cos x} + 2 \cos(k+1)\,y, \end{aligned}$$

we then get the three-term recurrence relation for $J_k(x)$

$$\begin{aligned} J_0(x) &= 0 \\ J_1(x) &= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(2+\alpha+\beta)} \\ J_{k+2}(x) - 2\cos x\, J_{k+1}(x) + J_k(x) &= d_{k+1}, \qquad k \geqslant 0, \end{aligned} \tag{3.6}$$

where the sequence $d_k$ is given by

$$\begin{aligned} d_k &= 2\int_0^{\pi} \sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2} \cos ky\, dy \\ &= 4\int_0^{\pi/2} \sin^{\bar{\alpha}} y \cos^{\bar{\beta}} y \cos 2ky\, dy, \qquad k = 1, 2, \dots . \end{aligned} \tag{3.7}$$

To compute the sequence $d_k$, we note that the following identity

$$\begin{aligned} \cos 2ky &= \tfrac{1}{2}\{(\cos y + i\sin y)^{2k} + (\cos y - i\sin y)^{2k}\} \\ &= \sum_{j=0}^{k} \binom{2k}{2j} (-1)^{k-j} \cos^{2j} y \sin^{2k-2j} y. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$d_k = 4\sum_{j=0}^{k} \binom{2k}{2j} (-1)^{k-j}\, C_{k-j,\,j}, \tag{3.9}$$

where the terms $C_{m,n}$ are defined by

$$C_{m,n} = \int_0^{\pi/2} \sin^{2m+\bar{\alpha}} y \cos^{2n+\bar{\beta}} y\, dy, \qquad m, n \geqslant 0, \tag{3.10}$$

which gives

$$\begin{aligned} C_{m,n} &= \tfrac{1}{2} B(m+\alpha+1, n+\beta+1) \\ &= \frac{\Gamma(m+\alpha+1)\,\Gamma(n+\beta+1)}{2\Gamma(m+n+\alpha+\beta+2)}, \end{aligned} \tag{3.11}$$

where $B(\cdot\,,\cdot)$ is the Euler beta function. Hence the sequence $d_k$ is given by

$$d_k = 2\sum_{j=0}^{k} \binom{2k}{2j} (-1)^{k-j} B(k-j+\alpha+1, j+\beta+1). \tag{3.12}$$

Using the quadrature scheme $Q_N(h; x)$ of (3.4), we now get a new integration formula

$$I_N(w(\cos\,\cdot\,)\,h; \cos x) = 2^{\alpha+\beta+1}\, Q_N(h; x) + h(x)\, q_0(x)$$
$$= 2^{\alpha+\beta+1} \sum_{k=0}^{N}{}'' a_k^N\, J_k(x) + f(\cos x)\, q_0(x), \qquad (3.13)$$

where $h(y) = f(\cos y)$ and the coefficients $a_k^N$ and $J_k(x)$ are given in (3.3) and (3.6), respectively, and $q_0(x)$ is given by (2.5).

In the rest of this section, we will describe an automatic quadrature algorithm of nonadaptive type which is constructed from the sequence of the truncated cosine series. Here and henceforth we assume that $N$ is of the form $2^n$; $n = 2, 3, \ldots$. Following the procedure of [10, 11, 17], we first outline the algorithm for computing the sequence of the truncated cosine series $\{p_N, p_{5N/4}, p_{3N/2}\}$.

For $\sigma = 2, 4$, set

$$v_j^{N/\sigma} = \begin{cases} 8\pi\left(j + \dfrac{3}{16}\right)\Big/N & \text{if } \sigma = 4 \\ 4\pi\left(j + \dfrac{3}{8}\right)\Big/N & \text{if } \sigma = 2. \end{cases}$$

Then we see that $\{v_j^{N/\sigma}\}$, $0 \leqslant j < N/\sigma$, is a set consisting of the $N/\sigma$ zeros of $\cos((N/\sigma)\, y) - \cos(3\pi/(2\sigma))$. We now represent the polynomials $p_{N+N/\sigma}$, $\sigma = 2, 4$, interpolating $h(y)$ at the nodes $v_j^{N/\sigma}$, $0 \leqslant j < N/\sigma$, $\sigma = 2, 4$, as well as at the nodes $\pi j/N$, $0 \leqslant j \leqslant N$, in the Newton form:

$$p_{N+N/\sigma}(y) - p_N(y)$$
$$= \sum_{k=1}^{N/\sigma} b_k^{N/\sigma}\, (\cos((N-k)\, y) - \cos((N+k)\, y)), \qquad (3.14)$$

where coefficients $\{b_k^{N/\sigma}\}$ are determined to satisfy the condition

$$h(v_j^{N/\sigma}) = p_{N+N/\sigma}(v_j^{N/\sigma}), \qquad 0 \leqslant j < \frac{N}{\sigma}, \quad \sigma = 2, 4. \qquad (3.15)$$

It must be noted that the interpolating conditions (3.15) are the same as those of the scheme of Hasegawa and Torii [7], and hence we can use the FFT [17] for evaluating the coefficients $b_k^{N/\sigma}$ (see [7, 17] for details).

Set $a_k^{N+N/\sigma}$, $\sigma=2, 4$, by

$$a_k^{N+N/\sigma}=\begin{cases} a_k^N, & 0\leqslant k<N-N/\sigma, \\ a_k^N+b_{N-k}^{N/\sigma}, & N-N/\sigma\leqslant k<N, \\ \dfrac{a_N^N}{2}, & k=N, \\ -b_{k-N}^{N/\sigma}, & N<k\leqslant N+N/\sigma. \end{cases} \tag{3.16}$$

Then $p_{N+N/\sigma}(y)$ of (3.14) can be written as

$$p_{N+N/\sigma}(y)=\sum_{k=0}^{N+N/\sigma}{}' \; a_k^{N+N/\sigma}\cos ky, \tag{3.17}$$

where the prime denotes the summation whose first term is halved.

The corresponding quadrature rules $I_{N+N/\sigma}(w(\cos\,\cdot\,)\,h;\cos x)$, $\sigma=2, 4$, based on the polynomial $p_{N+N/\sigma}(y)$ of (3.17) are given as follows,

$$I_{N+N/\sigma}(w(\cos\,\cdot\,)\,h;\cos x)=\sum_{k=0}^{N+N/\sigma}{}' \; a_k^{N+N/\sigma}\,J_k(x)+f(\cos x)\,q_0(x), \tag{3.18}$$

where $h(y)=f(\cos y)$ and $J_k(x)$ and $q_0(x)$ are given by (3.6) and (2.5), respectively.

## 4. ASYMPTOTIC BEHAVIOUR OF THE QUADRATURE WEIGHTS

In this section, we consider the asymptotic behaviour of the sequence $d_k$ of (3.7) and a related sequence with $J_k(x)$.

LEMMA 4.1. *Let $d_k$ be defined by* (3.7). *Let $\bar{\alpha}=2\alpha+1$, $\bar{\beta}=2\beta+1$, and $-\frac{1}{2}<\alpha, \beta<1$. Then we have the bounds*

$$|d_k|=O\left(\frac{1}{k}\right) \tag{4.1}$$

*for sufficiently large k.*

*Proof.* Replacing $y$ by $y+\frac{\pi}{2k}$ in the integral (3.7) and taking the mean value of the new and old integrals, we see that

$$\begin{aligned} d_k &= 4\int_0^{\pi/2}\sin^{\bar{\alpha}} y\cos^{\bar{\beta}} y\cos 2ky\,dy \\ &= -4\int_{-\pi/(2k)}^{\pi/2-\pi/(2k)}\sin^{\bar{\alpha}}\left(y+\frac{\pi}{2k}\right)\,\cos^{\bar{\beta}}\left(y+\frac{\pi}{2k}\right)\cos 2ky\,dy \\ &= 2(A_1+A_2+A_3), \end{aligned}$$

where

$$A_1 = \int_{\pi/2-h}^{\pi/2} \sin^{\bar{\alpha}} y \cos^{\bar{\beta}} y \cos 2ky \, dy,$$

$$A_2 = -\int_{-h}^{0} \sin^{\bar{\alpha}}(y+h) \cos^{\bar{\beta}}(y+h) \cos 2ky \, dy,$$

$$A_3 = \int_{0}^{\pi/2-h} [\sin^{\bar{\alpha}} y \cos^{\bar{\beta}} y - \sin^{\bar{\alpha}}(y+h) \cos^{\bar{\beta}}(y+h)] \cos 2ky \, dy,$$

$$h = \frac{\pi}{2k}.$$

The estimations of the integrals $A_1$ and $A_2$ can be achieved as follows.

$$\begin{aligned} |A_1| &\leqslant \int_{\pi/2-h}^{\pi/2} \sin^{\bar{\alpha}} y \cos^{\bar{\beta}} y \, dy \\ &= \int_{\pi/2-h}^{\pi/2} \cos^{\bar{\beta}} y \sin y \sin^{2\alpha} y \, dy \\ &\leqslant \max_{y \in [\pi/2-h, \pi/2]} \sin^{2\alpha} y \int_0^{\sin h} t^{\bar{\beta}} \, dt \\ &= \frac{\max_{y \in [\pi/2-h, \pi/2]} \sin^{2\alpha} y}{\bar{\beta}+1} \sin^{\bar{\beta}+1} h \\ &= O\left(\frac{1}{k^{2(\beta+1)}}\right), \end{aligned}$$

where we used the change of variables $t = \cos y$ in the second equality above. Similarly, by using the change of variables $t = \sin y$, we get $A_2 = O(1/k^{2(\alpha+1)})$.

By the triangle inequality and the fundamental theorem of calculus, we bound the integral $A_3$ as follows.

$$\begin{aligned} |A_3| &\leqslant \int_0^{\pi/2-h} |\sin^{\bar{\alpha}} y \cos^{\bar{\beta}} y - \sin^{\bar{\alpha}}(y+h) \cos^{\bar{\beta}}(y+h)| \, dy \\ &\leqslant |\bar{\alpha}| \, A_3^1 + |\bar{\beta}| \, A_3^2, \end{aligned}$$

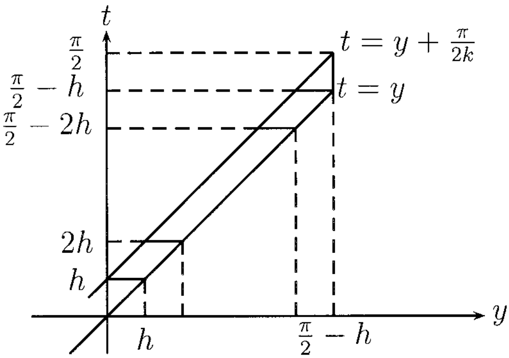


**FIG. 1.** The partition of the region of the integral $A_3^1$ with $h=\frac{\pi}{2k}$.

where

$$A_3^1=\int_0^{\pi/2-h}\int_y^{y+h}\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dt \, dy$$

$$A_3^2=\int_0^{\pi/2-h}\int_y^{y+h}\sin^{\bar{\alpha}}(y+h)\cos^{\bar{\beta}-1} t \sin t \, dt \, dy .$$

To estimate the integrals $A_3^1$ and $A_3^2$, we split the region of these integrals as in Fig. 1 and apply Fubini's theorem. We then get

$$A_3^1=D_1+D_2+D_3+D_4+D_5,$$

where

$$D_1=\int_0^h\int_0^t\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dy \, dt$$

$$D_2=\int_h^{2h}\int_{t-h}^t\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dy \, dt$$

$$D_3=\int_{\pi2-2h}^{\pi2-h}\int_{t-h}^t\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dy \, dt$$

$$D_4=\int_{\pi2-2h}^{\pi2-h}\int_{\pi2-h}^{y+h}\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dt \, dy$$

$$D_5=\int_{2h}^{\pi2-2h}\int_{t-h}^t\cos^{\bar{\beta}} y \sin^{\bar{\alpha}-1} t \cos t \, dy \, dt.$$

We now estimate each integral $D_i$ as follows.

$$D_1 \leqslant \max_{y \in [0,\, h]} \cos^{\bar{\beta}} y \int_0^h \int_0^t dy \sin^{\bar{\alpha}-1} t \cos t \, dt$$

$$\leqslant \max_{y \in [0,\, h]} \cos^{\bar{\beta}} y \int_0^h t \sin^{\bar{\alpha}-1} t \, dt$$

$$\leqslant C_{\bar{\alpha}} \max_{y \in [0, h]} \cos^{\bar{\beta}} y \int_0^h t^{\bar{\alpha}} \, dt$$

$$= \frac{C_{\bar{\alpha}} \max_{y \in [0,h]} \cos^{\bar{\alpha}} y}{\bar{\alpha}+1} h^{\bar{\alpha}+1}$$

$$= O\left(\frac{1}{k^{\bar{\alpha}+1}}\right),$$

where $C_{\bar{\alpha}} = 1$ if $\bar{\alpha} \geqslant 1$ and $C_{\bar{\alpha}} = (\frac{2}{\pi})^{\bar{\alpha}-1}$ if $\bar{\alpha} < 1$ and we used the inequality $\frac{2}{\pi} t < \sin t < t$ in the third inequality above. To estimate the integral $D_2$, we set the inner integral of $D_2$ as

$$F(t) = \int_{t\text{-}h}^{t} \cos^{\bar{\beta}} y \, dy.$$

Then we see that $F(t) = O(1/k)$, because $\cos^{\bar{\beta}} y \leqslant 1$ and $\bar{\beta} > 0$. Thus the bounds for $D_2$ are as follows.

$$D_2 = \int_h^{2h} F(t) \sin^{\bar{\alpha}-1} t \cos t \, dt$$

$$= O\left(\frac{1}{k}\right) \int_h^{2h} \sin^{\bar{\alpha}-1} t \cos t \, dt$$

$$= O\left(\frac{1}{k}\right) \frac{1}{\bar{\alpha}} (\sin^{\bar{\alpha}} 2h - \sin^{\bar{\alpha}} h)$$

$$= O\left(\frac{1}{k}\right) \frac{1}{\bar{\alpha}} (\sin^{\bar{\alpha}} 2h + \sin^{\bar{\alpha}} h)$$

$$= O\left(\frac{1}{k^{\bar{\alpha}+1}}\right),$$

where we used the inequality $\frac{2}{\pi} t \leqslant \sin t \leqslant t$ in the last inequality above. If $\beta \leqslant 0$,

$$\begin{aligned}\max_{t\in[\pi/2-2h,\,\pi/2-h]} \int_{t\text{-}h}^{t} \cos^{\bar{\beta}} y\,dy &\leqslant \int_{\pi/2-2h}^{\pi/2-h} \cos^{\bar{\beta}} y\,dy \\ &\leqslant \int_{\pi/2-2h}^{\pi/2} (1\text{-}\sin\,y)^{\beta}\,(1+\sin\,y)^{\beta} \cos\,y\,dy \\ &\leqslant \frac{\max_{y\in[\pi/2-2h,\,\pi/2]}(1+\sin\,y)^{\beta}}{\beta+1} \\ &\quad \times(1-\cos\,2h)^{\beta+1} \\ &= O\left(\frac{1}{k^{2(\beta+1)}}\right).\end{aligned}$$

For the case $\beta \geqslant 0$, the above procedure gives

$$\begin{aligned}\max_{t\in[\pi/2\text{-}2h,\,\pi/2\text{-}h]} \int_{t\text{-}h}^{t} \cos^{\bar{\beta}} y\,dy &\leqslant \int_{\pi/2\text{-}3h}^{\pi/2\text{-}2h} \cos^{\bar{\beta}} y\,dy \\ &= O\left(\frac{1}{k^{2(\beta+1)}}\right).\end{aligned}$$

Thus, we have that

$$\begin{aligned}D_3 &= \int_{\pi/2-2h}^{\pi/2-h} \int_{t\text{-}h}^{t} \cos^{\bar{\beta}} y\,dy\,\sin^{\bar{\alpha}-1} t\,\cos\,t\,dt \\ &\leqslant \max_{t\in[\pi/2-2h,\,\pi/2]} \sin^{\bar{\alpha}-1} t \max_{t\in[\pi/2-h,\,\pi/2\text{-}2h]} \int_{t\text{-}h}^{t} \cos^{\bar{\beta}} y\,dy \int_{\pi/2-2h}^{\pi/2-h} \cos\,t\,dt \\ &= O\left(\frac{1}{k^{2(\beta+1)}}\right).\end{aligned}$$

By the mean value theorem, we can estimate the integral $D_4$ as follows.

$$\begin{aligned}D_4 &= \int_{\pi/2-2h}^{\pi/2-h} \cos^{\bar{\beta}} y \int_{\pi/2\text{-}h}^{y+h} \sin^{\bar{\alpha}-1} t\,\cos\,t\,dt\,dy \\ &\leqslant \sin\,h \int_{\pi/2-2h}^{\pi/2-h} \cos^{\bar{\beta}} y\,dy \int_{\pi/2-h}^{\pi/2} \sin^{\bar{\alpha}-1} t\,dt \\ &= \sin\,h\; O\left(\frac{1}{k^{2\beta+1}}\right) \int_{\pi/2-h}^{\pi/2} \sin^{\bar{\alpha}-1} t\,dt \\ &= O\left(\frac{1}{k^{2(\beta+1)}}\right).\end{aligned}$$

To estimate the integral $D_5$, we also set the inner integral of $D_5$ as

$$F(t)=\int_{t-h}^{t} \cos^{\bar{\beta}} y\,dy.$$

Then we see that $F(t)\leqslant h$, because $\bar{\beta}>0$ and $\cos^{\bar{\beta}} y\leqslant 1$. Thus we have

$$\begin{aligned} D_5 &\leqslant h\int_{2h}^{\pi/2-2h} \sin^{\bar{\alpha}-1} t\cos t\,dt \\ &\leqslant \frac{h}{\bar{\alpha}}(\cos^{\bar{\alpha}} 2h-\sin^{\bar{\alpha}} 2h), \\ &= O(h). \end{aligned}$$

Collecting the bounds of each integral $D_i$, we then have

$$|A_3^1|=O\left(\frac{1}{k}\right),$$

since $\min(2(\alpha+1), 2(\beta+1), 1)=1$. By a similar procedure, we can get the bounds for $A_3^2$ as follows:

$$|A_3^2|=O\left(\frac{1}{k}\right).$$

Finally, we summarize the above estimations and get the desired bounds (4.1). ∎

Now, by using the identity [7, (A.3)],

$$\frac{\cos(k+1)\,y-\cos(k+1)\,x}{\cos y-\cos x}=2\sum_{j=0}^{k}{}' \frac{\sin(k-j+1)\,x}{\sin x}\cos jy, \qquad k\geqslant 0,$$

we can write $J_k(x)$ defined in (3.5) as

$$J_{k+1}(x)= 2\sum_{j=0}^{k}{}' \frac{\sin(k-j+1)\,x}{\sin x}\,d_j, \qquad k\geqslant 0, \tag{4.2}$$

where $d_j$ are defined in (3.7).

LEMMA 4.2. *Let $J_k(x)$ and $d_k$ be given in* (4.2) *and* (3.12), *respectively. Then we have*

$$J_{m-1}(x)-J_{m+1}(x)= -4\sum_{j=0}^{m}{}'' d_j\cos(m-j)\,x, \tag{4.3}$$

*where* $d_0 = J_1(x)$. *Further we have the following estimation*

$$|J_{m-1}(x) - J_{m+1}(x)| = O(1 + \log(m+1)). \tag{4.4}$$

*Proof.* The representation (4.3) follows directly from the expression (4.2) by defining $d_0 = J_1(x)$. The relation (4.3) and the bounds (4.1) give

$$\begin{aligned} |J_{m-1}(x) - J_{m+1}(x)| &\leqslant 4 \sum_{j=0}^{m} |d_j| \\ &\leqslant C(\alpha, \beta)\left(1 + \sum_{j=2}^{m} \frac{1}{j}\right) \\ &\leqslant C\left(1 + \int_0^m \frac{1}{x+1}\, dx\right) \\ &= O(1 + \log(m+1)). \end{aligned}$$

Thus we proved the estimation (4.4). ∎

## 5. ERROR ESTIMATES

In this section, we shall derive error bounds for the proposed quadrature rule of (3.13). Let $\varepsilon_\kappa$ denote the ellipse in the complex $z = x + iy$ with foci $(x, y) = (-1, 0), (1, 0)$, and semimajor axis $a = \frac{1}{2}(\kappa + \kappa^{-1})$ and semiminor axis $b = \frac{1}{2}(\kappa - \kappa^{-1})$ for a constant $\kappa > 1$.

Assume that $f(z)$ is single-valued and analytic inside and on $\varepsilon_\kappa$. Then, by (3.2) of [7], we have

$$\begin{aligned} h(x) - p_N(x) &= f(\cos x) - p_N(x) \\ &= \frac{1}{2\pi i} \oint_{\varepsilon_\kappa} \frac{\omega_{N+1}(\cos x)\ f(z)}{(z - \cos x)\ \omega_{N+1}(z)}\, dz \\ &= -2 \sum_{k=0}^{\infty}{}' V_k^N(f) \sin y \sin N y \cos ky, \end{aligned} \tag{5.1}$$

where $\omega_{N+1}(t) = T_{N+1}(t) - T_{N-1}(t) = 2(t^2 - 1)\, U_{N-1}(t)$, $N \geqslant 1$ and

$$V_k^N(f) = \frac{1}{\pi i} \oint_{\varepsilon_\kappa} \frac{f(z)}{\omega_{N+1}(z) \sqrt{z^2 - 1}\, (z + \sqrt{z^2 - 1})^k}\, dz, \qquad k \geqslant 0. \tag{5.2}$$

Here, $T_k(t)$ and $U_k(t)$ denote the Chebyshev polynomials of the first and second kind, respectively.

Using (5.1), we now define the error for the approximate integral $I_N(w(\cos\ \cdot\ )\,h; \cos x)$ as

$$E_N(h; x) = I(w(\cos\ \cdot\ )\,h; \cos x) - I_N(w(\cos\ \cdot\ )\,h; \cos x)$$

$$= -2^{\alpha+\beta+2} \sum_{k=0}^{\infty}{}' V_k^N(f)\, \Omega_k^N(x), \tag{5.3}$$

where $h(y) = f(\cos y)$ and

$$\Omega_k^N(x) = \int_0^\pi \sin^{\bar{\alpha}} \frac{y}{2} \cos^{\bar{\beta}} \frac{y}{2} \frac{\sin y \sin Ny \cos ky - \sin x \sin Nx \cos kx}{\cos y - \cos x}\, dy. \tag{5.4}$$

Then we have the following lemma.

LEMMA 5.1. (1) *Let $\Omega_k^N(x)$ be defined by (5.4). Then $\Omega_k^N(x)$ can be rewritten as*

$$\Omega_k^N(x) = \tfrac{1}{4}\,(J_{N+k-1}(x) - J_{N+k+1}(x) \pm (J_{|N-k|-1}(x) - J_{|N-k|+1}(x)), \tag{5.5}$$

*where the plus sign is taken if $N-k \geqslant 1$ and the minus sign if $N-k<1$, and $J_k(x)$, $k \geqslant 0$, are given by (3.6).*

(2) *Further, $\Omega_k^N(x)$ are bounded by*

$$|\Omega_k^N(x)| \leqslant CG(N, k), \qquad k \geqslant 0, \tag{5.6}$$

*where $C = C(\alpha, \beta)$ is a constant dependent of $\alpha$, $\beta$ and $G(N, k)$ is defined by*

$$G(N, k) = 1 + \log((N+k+1)(|N-k|+1)). \tag{5.7}$$

*Proof.* The first identity (5.5) is obtained by the definition of $J_k(x)$ of (3.5) and the relation

$$4 \sin y \sin Ny \cos ky = \cos(N+k-1)\,y - \cos(N+k+1)\,y$$
$$+ \cos(N-k-1)\,y - \cos(N-k+1)\,y.$$

The bound (5.6) follows from inequality (4.4) of Lemma 4.3 and identity (5.5). ∎

From (5.3) and (5.6), we have the following lemma.

LEMMA 5.2. *Let $N=2^n$, $n=2, 3, \ldots$, and assume that $f(z)$ is single-valued and analytic inside and on $\varepsilon_\rho$. Let $h(y)=f(\cos y)$. Then the error $E_N(h; x)$ of* (5.3) *is bounded independent of $x$ by*

$$|E_N(h; x)| \leqslant C \sum_{k=0}^{\infty}{}' \, G(N, k) \, |V_k^N(f)|, \tag{5.8}$$

*where $V_k^N(f)$ and $G(N, k)$ are given by* (5.2) *and* (5.7), *respectively, and the constants $C=C(\alpha, \beta)$ are dependent of $\alpha$ and $\beta$.*

For the function $h(y)=f(\cos y)$, we define

$$\begin{aligned} E_{N+N/\sigma}(h; x) := {} & I(w(\cos\ \cdot\ )\, h; \cos x) \\ & - I_{N+N/\sigma}(w(\cos\ \cdot\ )\, h; \cos x), \end{aligned} \tag{5.9}$$

where $\sigma=2, 4$. Then, by the same arguments as in [7], and the previous results, we get the following error bounds for $E_{N+N/\sigma}(h; x)$.

LEMMA 5.3. *Let $N=2^n$, $n=2, 3, \ldots$, and assume that $f(z)$ is single valued and analytic inside and on $\varepsilon_\rho$. Further, let $V_k^{N+N/\sigma}(f)$, $\sigma=2, 4$, denote a contour integral defined by*

$$V_k^{N+N/\sigma}(f) = \frac{1}{\pi i} \oint_{\varepsilon_\rho} \frac{f(z)\, dz}{\omega_{N+1}(z)\{T_{N/\sigma}(z) - \cos 2\pi\beta_\sigma\} \sqrt{z^2-1}\, w^k}, \qquad k \geqslant 0, \tag{5.10}$$

*where $w=z+\sqrt{z^2-1}$ and $|w|>1$ if $z \notin [-1, 1]$. Then, for $h(y)=f(\cos y)$, we have*

$$\begin{aligned} |E_{N+N/\sigma}(h; x)| \leqslant C \sum_{k=0}^{\infty}{}' \, |V_k^{N+N/\sigma}(f)| \Big( & |\cos 2\pi\beta_\sigma| \, G(N, k) \\ & + \left( G\left(N+\frac{N}{\sigma}, k\right) + G\left(N-\frac{N}{\sigma}, k\right)\right)\Big), \end{aligned} \tag{5.11}$$

*where $\beta_4=3/16$ and $\beta_2=3/8$.*

Finally, to estimate the bounds of $V_k^N(f)$ of (5.2) and $V_k^{N+N/\sigma}(f)$ of (5.10), we follow the procedure of [7, 11]. Suppose that $f(z)$ is a meromorphic function which has $M$ simple poles at the points $z_m$, $m=1, 2, \ldots, M$, outside $\varepsilon_\rho$ with residues $Res\, f(z_m)$. Define

$$r = \min_{1 \leqslant m \leqslant M} |z_m + \sqrt{z_m^2 - 1}| > 1. \tag{5.12}$$

Then, from [11, (3.36) and (3.37)], (5.2) and (5.10), we see that $\kappa < r$ and

$$|V_k^N(f)| \leqslant r^{-k} |V_0^N(f)| = O(r^{-k-N}) \tag{5.13}$$

and

$$|V_k^{N+N/\sigma}(f)| \leqslant r^{-k} |V_0^{N+N/\sigma}(f)| = O(r^{-k-N-N/\sigma}). \tag{5.14}$$

Further, from [7, (3.17) and (3.20)], we can see that

$$|V_0^N(f)| \sim \frac{r}{r^2-1} |a_N^N| \tag{5.15}$$

and

$$|V_0^{N+N/\sigma}(f)| \sim |a_{N+N/\sigma}^{N+N/\sigma}| \frac{4r}{r^2-1}, \tag{5.16}$$

where $a_N^N/2$ and $a_{N+N/\sigma}^{N+N/\sigma}$ are the coefficients of the last term in the truncated cosine series (3.1) and (3.17), respectively. The constant $r$ defined in (5.12) can be estimated from the asymptotic behavior of $\{a_k^N\}$ [17].

Define a function $H(n, r)$ by

$$H(n, r) = \frac{r+1}{r-1} \log(n+1)\, e^{1/2} + 2\,\frac{(n+2)(r-1)-1}{r^n(r-1)^2}. \tag{5.17}$$

Then we see that

$$H(n, r) = O(1 + \log(n+1)).$$

By using (5.13)–(5.16) and Lemmas 5.2–5.3, we obtain the following error estimations.

THEOREM 5.4. *Let* $N = 2^n$, $n = 2, 3, \ldots,$ *and suppose that* $f(z)$ *is a meromorphic function which has* $M$ *simple poles at the points* $z_m$, $m = 1, 2, \ldots, M$ *outside* $\varepsilon_\rho$ *with residues* $\operatorname{Res} f(z_m)$. *Let* $r = \min_{1 \leqslant m \leqslant M} |z_m + \sqrt{z_m^2 - 1}| > 1$ *and* $h(y) = f(\cos y)$. *Then the errors* $E_N(h; x)$ *and* $E_{N+N/\sigma}(h; x)$, $\sigma = 2, 4$, *defined in* (5.3) *and* (5.9), *respectively, can be bounded independent of* $x$ *as follows*;

$$|E_N(h; x)| \leqslant C \frac{r}{r^2-1} |a_N^N|\, H(N, r) \tag{5.18}$$

*and*

$$|E_{N+N/\sigma}(h;x)| \leqslant C\frac{4r}{r^2-1}\,|a_{N+N/\sigma}^{N+N/\sigma}|\,(H(N+N/\sigma,r)$$
$$+H(N-N/\sigma,r)+|\cos 2\pi\beta_\sigma|\,H(N,r)), \qquad (5.19)$$

*where* $H(n,r)$ *are given by* (5.17) *and the constants* $C$ *are dependent of* $\alpha$ *and* $\beta$.

*Proof.* Substituting the inequality (5.13) into (5.8) yields

$$|E_N(h;x)| \leqslant C\,|V_0(f)|\,{\sum_{k=0}^{\infty}}'\,\frac{G(N,k)}{r^k}$$
$$\leqslant |V_0(f)|\left({\sum_{k=0}^{N}}'\,\frac{G(N,0)}{r^k}+2\sum_{k=N+1}^{\infty}\frac{k+1}{r^k}\right), \qquad (5.20)$$

where we used the facts that the function $G(N,k)$ of (5.7) satisfies

$$G(N,k) \leqslant G(N,0), \qquad 0 \leqslant k \leqslant N$$

and

$$G(N,k) \leqslant 2(k+1), \qquad k=N+1, N+2, \ldots.$$

Since

$$
{\sum_{k=0}^{N}}'\,\frac{1}{r^k} \leqslant \frac{r+1}{2(r-1)} \qquad (5.21)
$$

and

$$
\sum_{k=N+1}^{\infty}\frac{k+1}{r^k}=\frac{(N+2)(r-1)-1}{r^N(r-1)^2}, \qquad (5.22)
$$

we substitute (5.21) and (5.22) into (5.20) and then get

$$|E_N(h;x)| \leqslant C\,|V_0(f)|\,H(N,r),$$

where $H(n,r)$ are given by (5.17). Thus from the asymptotic behaviour of $V_0(f)$ given in (5.15), we get the desired inequality (5.18). By a procedure similar to that above and using (5.14) and (5.16), we can get the required inequality (5.19). ∎

We remark that the error estimates (5.18) and (5.19) for the quadrature rules $E_N(h; x)$ and $E_{N+N/\sigma}(h; x)$, respectively, are independent of the values of poles $x$. This fact enables us to use the same polynomials $p_N(y)$ and $p_{N+N/\sigma}(y)$ common to the integrals $I(w(\cos\,\cdot\,)\,h; \cos x)$ for a set of values of poles $x$.

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